Probability and Expected Value

This handout provides an introduction to *probability* and *expected value*. Some of you may already be familiar with some of these topics. *Probability* and *expected value* are used in statistics, finance, economics and elsewhere.

These topics are introduced here with some very easy examples. The theory is then extended to consider some more common applications, such as what are your chances of winning the lottery, or winning at Casino Niagara. We will then use these concepts in this course to discuss insurance and simple asset pricing. You will also use these concepts in finance courses to evaluate various investments.

Introduction to Probability and Expected Value

Probability is related to the frequency that an event is predicted to occur. For example, if you flip a coin in the air 100 times, the coin will land "heads-up" (that is, with the picture of the Queen face-up) approximately half the time. The rest of the time, the coin will land "tails-up"

Probabilities are assigned a number between zero and one. A probability of one indicates that the event will occur with complete certainty; a probability of zero indicates an event that will not occur. In the coin flipping example, we can denote the probability of the coin landing heads-up as p_H and the probability of the coins landing tails-up as p_T

Since we would expect that a coin is just as likely to land "heads" as "tails", we write:
$p_H = \frac{1}{2}$, $p_T = \frac{1}{2}$

Another simple example of probability can be found in rolling dice. A standard 'die' has six sides. Each side has either one, two, three, four, five or six dots, representing the numbers 1, 2, 3, 4, 5, 6. We can think of these as being 6 different *outcomes* or *events*. If we assume the die is perfectly balanced, the probability of any particular outcome (say, rolling a '3') is 1 out of 6. Therefore, in this example, we could write:

$p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = \frac{1}{6}$

where $p_1 \equiv$ probability of rolling a 1, $p_2 \equiv$ probability of rolling a 2, etc ...

Notice that in both examples, the sum of the probabilities add up to one.
In the coin flipping example,: $p_H + p_T = \frac{1}{2} + \frac{1}{2} = 1.$
When rolling a die: $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$

This makes sense, because we know that when flipping the coin, the coin will land either "heads" or "tails" (we will ignore the possibility of the coin landing on its edge). That is, we are certain that either “heads” or “tails” will be the outcome. And events that will happen with certainty are assigned a probability of 1.

Similarly, when we roll a die, the die will land showing the side with either 1, 2, 3, 4, 5, or 6 dots face up. Although it is not certain which number will turn up, we know that each time we roll the die, one of the six outcomes will occur. So the sum of the six outcomes ($\boldsymbol{p_1} + \boldsymbol{p_2} + \boldsymbol{p_3} + \boldsymbol{p_4} + \boldsymbol{p_5} + \boldsymbol{p_6}$) must be 1.

Assigning values to outcomes

In economics, it is useful to consider cases where there are different economic outcomes associated with a specific event. For example, suppose that you ask your father for some extra money. Instead of just giving you the money, he offers to flip a coin. If the coin lands “heads” – you win – and he will give you $20. If it lands “tails”, you lose, and you receive nothing. In this example, we would say that the ***payout*** associated with the event that the coin lands “heads” is $20, and the payout associated with “tails” is $0.

Before accepting your dad’s offer, you turn to your mother and ask her for some extra money. Having overheard the conversation with your father, she makes a different offer. She will role a die, and give you $3 for every spot that turns up. (That is, if she rolls a 1, she will give you $3; if she rolls a 2, she will give you $6.00 (= 2 x $3.00), ... if she rolls a 6, she will give you $18 (= 6 x $3.00)).

At this point you realize two things: First, you realize your parents are both very weird. Second, (since your mother and father won’t both give you money) you realize that you need some way to compare the two offers.

So which of these offers should you take? Which is the better offer?

We need some method of determining the amount of money that you could potentially make in each case. Then we need to adjust this for the likelihood of each particular outcome.

In economics and finance, the most common way of evaluating these alternatives is to calculate the ***expected value***. The ***expected value*** is the average of all possible outcomes, adjusted (or “weighted”) for the likelihood that each outcome will occur.

Consider your father’s coin flipping offer. We can use a variable, such as **X** to indicate the possible payouts. There are two possible outcomes, “heads” and “tails”. The payout you receive if the coin lands “heads” is $20, and we write: $\mathbf{X_H}$ = $20. Similarly, if the coin lands “tails” you get nothing, and we write: $\mathbf{X_T}$ = $0.

In this simple case, the expected value is given by the equation:

$$E(\mathbf{X}) = (\, p_H \times \mathbf{X_H}\,) + (\, p_T \times \mathbf{X_T}\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = (\, \tfrac{1}{2} \times 20\,) + (\, \tfrac{1}{2} \times 0\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = (\, 10\,) + (\, 0\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = \$10.00$$

(Note: $E(\mathbf{X})$ is notation for "the expected value of X". Of course, you will not actually receive a $10 bill from your father; you will receive either $20 or nothing. Hence – if you take each word literally – the phrase "expected value" is a bit misleading. Nonetheless, this is the terminology that is used.)

Now let's evaluate your mother's offer. She has offered to roll a die and pay you $3 times the number that comes up. We will use the variables $\mathbf{X_1}$, $\mathbf{X_2}$, $\mathbf{X_3}$, etc.., to represent the payouts associated with each number of the die (1,2,3, ...). So, the expected value here is:

$$E(\mathbf{X}) = (\, p_1 \times \mathbf{X_1}\,) + (\, p_2 \times \mathbf{X_2}\,) + (\, p_3 \times \mathbf{X_3}\,) + (\, p_4 \times \mathbf{X_4}\,) + (\, p_5 \times \mathbf{X_5}\,) + (\, p_6 \times \mathbf{X_6}\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = (\, \tfrac{1}{6} \times \mathbf{X_1}\,) + (\, \tfrac{1}{6} \times \mathbf{X_2}\,) + (\, \tfrac{1}{6} \times \mathbf{X_3}\,) + (\, \tfrac{1}{6} \times \mathbf{X_4}\,) + (\, \tfrac{1}{6} \times \mathbf{X_5}\,) + (\, \tfrac{1}{6} \times \mathbf{X_6}\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = (\, \tfrac{1}{6} \times 3\,) + (\, \tfrac{1}{6} \times 6\,) + (\, \tfrac{1}{6} \times 9\,) + (\, \tfrac{1}{6} \times 12\,) + (\, \tfrac{1}{6} \times 15\,) + (\, \tfrac{1}{6} \times 18\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = \tfrac{1}{6} \times (3 + 6 + 9 + 12 + 15 + 18\,)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = \tfrac{1}{6} \times (63)$$
$$\Leftrightarrow \quad E(\mathbf{X}) = \$10.50$$

Comparing the two offers, your mother's offer appears to be slightly better based on expected value. Of course, you may want to take a chance and try for the $20 bill being offered by your father. We will consider preferences between these two offers when we discuss *expected utility* in class. For now, we will simply note that the expected value of your mother's offer is slightly higher than the expected value of your father's offer.

Expected value is most useful in circumstances where you have an opportunity to repeatedly make the same decision. In the example above, the difference in expected value is only $0.50. However, if this was a daily occurrence, the extra value offered by your mother ($0.50) would start to add up.

Example: Expected Value and a Lottery

Let's consider some other instances where we can use the concept of expected value. Suppose that in order to raise money for a local seniors citizens home, the town council for Pickering decides to hold a charity lottery:

Enter the Charity Lottery
One Grand Prize of $20,000
20 additional prizes of $500

Tickets only $10

After reading the fine print, you discover that 10,000 tickets will be sold. Is this a good bet?

This example is almost the same as the previous example. The only differences being that the probability of the outcomes are not identical, and you have to buy a ticket to enter.

Setting aside the price of the ticket for the moment, if you are lucky enough to win the grand prize, your payout is:

$$\mathbf{X}_{\mathbf{WIN}} \equiv \$20{,}000$$

If you win one of the other (2nd place) prizes, your payout is:

$$\mathbf{X}_{\mathbf{2ND}} \equiv \$500$$

And if you don't win any prize, your payout is:

$$\mathbf{X}_{\mathbf{LOSE}} \equiv \$0$$

If you purchase a ticket, your expected winnings in the lottery (not including the ticket price) can be described by the following equation:

$$E(\mathbf{X}) = (\, p_{\mathbf{WIN}} \times \mathbf{X}_{\mathbf{WIN}}\,) + (\, p_{\mathbf{2ND}} \times \mathbf{X}_{\mathbf{2ND}}\,) + (\, p_{\mathbf{LOSE}} \times \mathbf{X}_{\mathbf{LOSE}}\,)$$

We already know the value of the payouts $\mathbf{X}_{\mathbf{WIN}}$, $\mathbf{X}_{\mathbf{2ND}}$ and $\mathbf{X}_{\mathbf{LOSE}}$, so now we simply have to determine the probabilities.

Since there is only one grand prize winner, and there are 10000 tickets being sold, the probability of winning the grand prize is: $p_{\mathbf{WIN}} = (1 / 10000) = 0.0001$

The probability of winning one of the second prizes is:

$$p_{\mathbf{2ND}} = (20 / 10000) = 0.002$$

Finally, since the probabilities must add up to one, we know that:

$$p_{\mathbf{WIN}} + p_{\mathbf{2ND}} + p_{\mathbf{LOSE}} = 1$$
$$\Leftrightarrow \quad p_{\mathbf{LOSE}} = 1 - p_{\mathbf{WIN}} - p_{\mathbf{2ND}}$$
$$\Leftrightarrow \quad p_{\mathbf{LOSE}} = 1 - 0.0001 - 0.002$$
$$\Leftrightarrow \quad p_{\mathbf{LOSE}} = 0.9979$$

Therefore, the expected winnings are given by the following expected value equation:

$E(\mathbf{X}) = (\boldsymbol{p}_{\mathbf{WIN}} \times \mathbf{X}_{\mathbf{WIN}}) + (\boldsymbol{p}_{\mathbf{2ND}} \times \mathbf{X}_{\mathbf{2ND}}) + (\boldsymbol{p}_{\mathbf{LOSE}} \times \mathbf{X}_{\mathbf{LOSE}})$

$\Rightarrow \quad E(\mathbf{X}) = (0.0001 \times \mathbf{X}_{\mathbf{WIN}}) + (0.002 \times \mathbf{X}_{\mathbf{2ND}}) + (0.9979 \times \mathbf{X}_{\mathbf{LOSE}})$

$\Leftrightarrow \quad E(\mathbf{X}) = (0.0001 \times 20000) + (0.002 \times 500) + (0.9979 \times 0)$

$\Leftrightarrow \quad E(\mathbf{X}) = (2.00) + (1.00) + (0)$

$\Leftrightarrow \quad E(\mathbf{X}) = \$ 3.00$

Once we include the $10 price of the ticket, the net expected value is then $3.00 - $10.00 = - $ 7.00. That is, ticket holders will lose $7.00 on average. (Of course, this lottery is for charity, so maybe losing $7.00 is OK).

Definitions and Notation

This section formalizes some of the ideas that have been presented above.

A variable that takes on different values based on chance is known as a *random variable*. A random variable is described by a set of possible *outcomes* and the *probabilities* associated with each outcome. Usually the random variable is represented by a simple letter (e.g. X), and the possible outcomes are represented by the same letter with an appropriate subscript. (e.g. $X_1, X_2, X_3, ...$).

Each outcome has an associated probability. Probabilities are usually represented by the letter “p” with the subscripts corresponding to the outcomes. (e.g. p_1 is the probability of X_1 occurring, p_2 is the probability of X_2 occurring, ...)

For a random variable X that has three possible outcomes $\{X_1, X_2, X_3\}$ with corresponding probabilities, p_1, p_2 and p_3, the *expected value* of X is defined as:

$$E(X) \equiv p_1X_1 + p_2X_2 + p_3X_3$$

If X has k possible outcomes, the *expected value* is defined as:

$$E(X) \equiv \sum_{(i = 1 .. k)} p_iX_i$$

For the random variable to be properly defined, the sum of the probabilities must equal 1 . For instance, if X has 3 possible outcomes and the associated probabilities are p_1, p_2 and p_3, then:

$$p_1 + p_2 + p_3 = 1 .$$

In the general case where X has *k* possible outcomes, this is written as:

$$\sum_{(i = 1 .. k)} p_i = 1$$

A function of a random variable is itself a random variable. Hence, if X is a random variable with possible outcomes $\{X_1, X_2, X_3\}$, and f(X) is a function of X, then

$$E(f(X)) \equiv p_1 f(X_1) + p_2 f(X_2) + p_3 f(X_3)$$

For example, suppose X is a random variable with three possible outcomes $\{X_1, X_2, X_3\}$. Then the function $f(X) \equiv X + 2$ is a random variable, and the expression for its expected value is:

$$E(X + 2) \equiv p_1 (X_1 + 2) + p_2 (X_2 + 2) + p_3 (X_3 + 2)$$

Similarly, the expected value equations for $f(X) \equiv 6X$ or $f(X) \equiv 4X^2 - 5$ are

$$E(6X) \equiv p_1(6X_1) + p_2(6X_2) + p_3(6X_3)$$

$$E(4X^2 - 5) \equiv p_1(4X_1^2 - 5) + p_2(4X_2^2 - 5) + p_3(4X_3^2 - 5)$$

Another useful property of expected value calculations is that the expected value of a constant is simply that constant. For example:

$E(3) = 3$
$E(19) = 19$
$E(c) = c$ (where c is a constant).

This leads to two simple theorems:

For any random variable X and any constant c :

1) $E(cX) = c \bullet E(X)$ (where "•" indicates multiplication)

and 2) $E(X + c) = E(X) + c$

Using this we can show that in the lottery example from above, the expected winnings can be calculated in two ways. If we use z to represent the ticket price, then we can either

a) calculate the expected value of the payouts, and then subtract the ticket price.

or

b) subtract the ticket price from each individual payout and then calculate the expected value.

That is, $E(X - z) = E(X) - z$

Proof: Let $z \equiv$ *ticket price*. Notice z is the same no matter what the outcome.

$$E(\mathbf{X - z}) = (\boldsymbol{p}_{\mathbf{WIN}}(\mathbf{X}_{\mathbf{WIN}} - \mathbf{z})) + (\boldsymbol{p}_{\mathbf{2ND}}(\mathbf{X}_{\mathbf{2ND}} - \mathbf{z})) + (\boldsymbol{p}_{\mathbf{LOSE}}(\mathbf{X}_{\mathbf{LOSE}} - \mathbf{z}))$$

$$\Leftrightarrow \quad E(\mathbf{X - z}) = (\boldsymbol{p}_{\mathbf{WIN}}(\mathbf{X}_{\mathbf{WIN}}) - \boldsymbol{p}_{\mathbf{WIN}}\mathbf{z}) + (\boldsymbol{p}_{\mathbf{2ND}}(\mathbf{X}_{\mathbf{2ND}}) - \boldsymbol{p}_{\mathbf{2ND}}\mathbf{z}) + (\boldsymbol{p}_{\mathbf{LOSE}}(\mathbf{X}_{\mathbf{LOSE}}) - \boldsymbol{p}_{\mathbf{LOSE}}\mathbf{z})$$

$$\Leftrightarrow \quad E(\mathbf{X - z}) = (\boldsymbol{p}_{\mathbf{WIN}}(\mathbf{X}_{\mathbf{WIN}}) + \boldsymbol{p}_{\mathbf{2ND}}(\mathbf{X}_{\mathbf{2ND}}) + \boldsymbol{p}_{\mathbf{LOSE}}(\mathbf{X}_{\mathbf{LOSE}})) - (\boldsymbol{p}_{\mathbf{WIN}}\mathbf{z} + \boldsymbol{p}_{\mathbf{2ND}}\mathbf{z} + \boldsymbol{p}_{\mathbf{LOSE}}\mathbf{z})$$

$$\Leftrightarrow \quad E(\mathbf{X - z}) = (\boldsymbol{p}_{\mathbf{WIN}}(\mathbf{X}_{\mathbf{WIN}}) + \boldsymbol{p}_{\mathbf{2ND}}(\mathbf{X}_{\mathbf{2ND}}) + \boldsymbol{p}_{\mathbf{LOSE}}(\mathbf{X}_{\mathbf{LOSE}})) - (\boldsymbol{p}_{\mathbf{WIN}} + \boldsymbol{p}_{\mathbf{2ND}} + \boldsymbol{p}_{\mathbf{LOSE}})\,\mathbf{z} \qquad (1)$$

But the first term of the right-hand-side of equation *(1)* is simply E(X)

$$\Leftrightarrow \quad E(\mathbf{X - z}) = E(\mathbf{X}) - (\boldsymbol{p}_{\mathbf{WIN}} + \boldsymbol{p}_{\mathbf{2ND}} + \boldsymbol{p}_{\mathbf{LOSE}})\,\mathbf{z} \qquad (2)$$

Also, we know that the sum of the probabilities of all possible outcomes is 1, hence

$$\Leftrightarrow \quad E(\mathbf{X - z}) = E(\mathbf{X}) - (\mathbf{1})\,\mathbf{z}$$ By substituting $\boldsymbol{p}_{\mathbf{WIN}} + \boldsymbol{p}_{\mathbf{2ND}} + \boldsymbol{p}_{\mathbf{LOSE}} = \mathbf{1}$ into *(2)*

$$\Leftrightarrow \quad E(\mathbf{X - z}) = E(\mathbf{X}) - \mathbf{z}$$

Therefore it doesn't matter if we subtract the ticket price "z" from each individual payout and then calculate the expected value or if we first calculate the expected value of the payouts and then subtract the ticket price.

(In order to differentiate between E(X) and E(X - z) in conversation, we can refer to E(X) as the *expected winnings* whereas we can refer to E(X - z) as the *net expected value* (that is, the expected winnings less the ticket price.))

Returning to the lottery example from above, we could have subtracted the price of the ticket from each of the payouts before calculating the expected value. That is, if we denote the ticket price as z = $10, we can calculate the expected value of (X - z). In this formulation, the outcomes are:

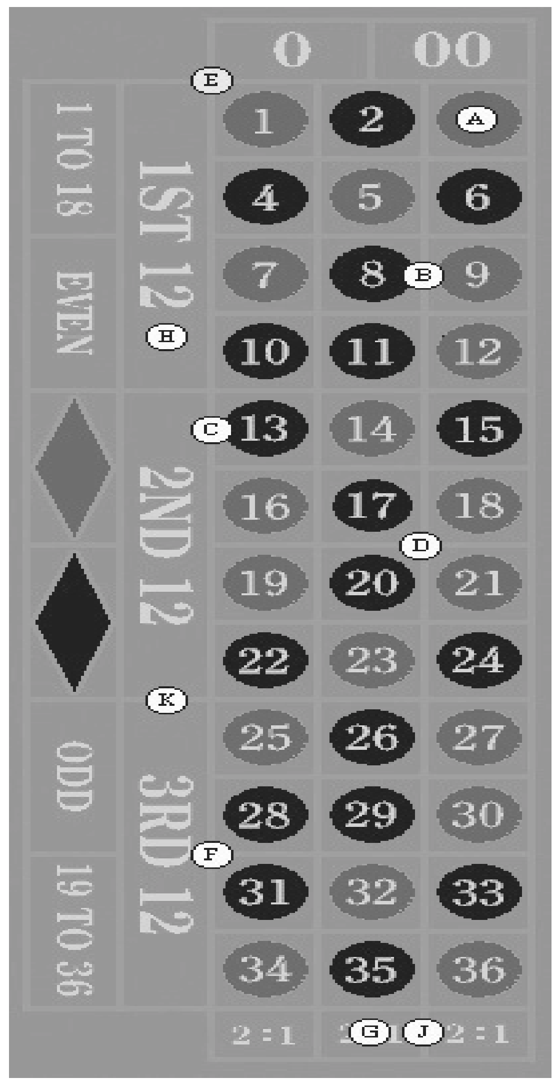
$$\mathbf{X_{WIN}} - \mathbf{z} \equiv 20000 - 10 \quad = \quad \$\ 19{,}990$$
$$\mathbf{X_{2ND}} - \mathbf{z} \equiv 500 - 10 \quad = \quad \$\ 490$$
$$\mathbf{X_{LOSE}} - \mathbf{z} \equiv 0 - 10 \quad = -\ \$\ 10$$

Then, the net expected value would be

$$E(\mathbf{X} - \mathbf{Z}) = (\, \boldsymbol{p}_{\mathbf{WIN}} \times (\mathbf{X_{WIN}} - \mathbf{z})) + (\, \boldsymbol{p}_{\mathbf{2ND}} \times (\mathbf{X_{2ND}} - \mathbf{z})) + (\, \boldsymbol{p}_{\mathbf{LOSE}} \times (\mathbf{X_{LOSE}} - \mathbf{z}))$$
$$\Rightarrow \quad E(\mathbf{X} - \mathbf{Z}) = (\, 0.0001 \times (\mathbf{X_{WIN}} - \mathbf{z})) + (\, 0.002 \times (\mathbf{X_{2ND}} - \mathbf{z})) + (\, 0.9979 \times (\mathbf{X_{LOSE}} - \mathbf{z}))$$
$$\Leftrightarrow \quad E(\mathbf{X} - \mathbf{Z}) = (\, 0.0001 \times 19990) + (\, 0.002 \times 490) + (\, 0.9979 \times -10\,)$$
$$\Leftrightarrow \quad E(\mathbf{X} - \mathbf{Z}) = (1.999) + (0.98) + (-9.979)$$
$$\Leftrightarrow \quad E(\mathbf{X} - \mathbf{Z}) = -\ \$\ 7.00$$

Which is the same answer as before. Generally, which formulation is used does not matter, although some problems may be better suited for one form than the other.

Casino Games of Chance



Although it may not be apparent at first, the games of chance offered in gambling casinos can be analysed using expected value calculations. We will take a look at *roulette*, which is one of the simpler casino games that you can play.

Roulette consists of a giant spinning wheel and a small silver ball. The roulette wheel consists of the numbers 1 through 36 plus 0 and "00" (known as double zero). Hence, there are a total of 38 possible outcomes.

Players bet on the outcome by placing chips (representing money) on a table. A casino employee then drops a ball onto the spinning wheel and everyone waits to see on which number the ball will land. An example of a roulette wheel and the table used for placing bets are shown at the right.

For example, a player who wants to bet that the roulette ball will land on the number 3 places a stack of chips on the table directly on the number 3 as indicated by point "A" at the right. If the ball lands on number 3, the player wins. Otherwise, the player loses.

Playing one number at a time is known as a *straight-up* bet. Casinos also allow players to bet on more than one number at a time by placing their chips in specific positions. For example, if a player places their chips at point "B" in the diagram at right, they are betting that the roulette ball will land on either the number 8 or the number 9. This is known as a *split* bet. In this case, the player will win if the ball lands on either the number 8 or the number 9. Of course, since a *split* bet has more chances of winning, the payout for a *split* bet is smaller than the payout for a *straight-up* bet.

There are many different types of bets available in roulette. The following chart summarizes these bets and their payouts:

Bet	Chances to Win	Net Payout	Example
"Straight-up"	1 number	35:1	Point A
"Split"	2 numbers	17:1	Point B
"3 Line"	3 numbers	11:1	Point C
"Corner"	4 numbers	8:1	Point D
"1st Five"	5 numbers	6:1	Point E
"6 Line"	6 numbers	5:1	Point F
"Column"	12 numbers	2:1	Point G
"Dozen"	12 numbers	2:1	Point H
"Split Columns"	24 numbers	1:2	Point J
"Split Dozens"	24 numbers	1:2	Point K
"Even/Odd"	18 numbers	1:1	
"Red/Black"	18 numbers	1:1	
"1-18/19-36"	18 numbers	1:1	

Note: the possible bets and their payouts can vary between casinos.

Since which number the ball lands on is completely random, the probability of "hitting" any particular number is 1 in 38. Consider some of the bets that can be made in Roulette.

Players can bet *straight-up* on each individual number (1 – 36, 0 and "00"). Suppose a player bets $10 straight-up on the number 3. The net payout ratio for a straight-up bet is 35 to 1. This means that if the ball lands on the number 3, the player receives $350 (= $360 - $10 initial bet). If the ball lands on any other number, the player loses. Since the payout for any number except the number 3 is the same, we can group these losing outcomes together. So, the net expected value of betting $10 straight-up on the number 3 is:

$$E(\mathbf{X}_{\text{STRAIGHT-UP}} - z) = P_3\,(X_3 - z) + P_{\text{LOSE}}\,(X_{\text{LOSE}} - z)$$
$$= (1/38) \times (360 - 10) + (37/38) \times (0 - 10) \qquad = - \$0.53$$
(= - 5.3 % of the initial $10 bet)

Players can also play a *corner* which covers four numbers. For a corner bet, the player wins if any of the four numbers "hit". The net payout for a winning corner bet is 8 to 1. Hence, the net expected value of betting $10 on a corner is:

$$E(\mathbf{X}_{\text{CORNER}} - z) = P_{\text{CORNER}}\,(X_{\text{CORNER}} - z) + P_{\text{LOSE}}\,(X_{\text{LOSE}} - z)$$
$$= (4/38) \times 80 + (34/38) \times -10 \qquad = - \$0.53$$
(= - 5.3 % of the initial $10 bet)

Alternatively, players may place a bet on all *odd* numbers. For this bet, a player wins if the ball lands on any odd number. The chances of winning by betting on odd numbers are much better than a *straight-up* bet or a *corner* bet. However, the net payout is only 1 to 1. Since there are 18 odd numbers, the net expected value of betting $10 on "odd" is:

$$E(\mathbf{X}_{\text{ODD}} - z) = P_{\text{ODD}}\,(X_{\text{ODD}} - z) + P_{\text{LOSE}}\,(X_{\text{LOSE}} - z)$$
$$= (18/38) \times 10 + (20/38) \times -10 \qquad = - \$0.53$$
(= - 5.3 % of the initial $10 bet)

So far, the various bets in roulette seem to be pretty much the same. Each has had a net expected return of - 5.3 %. Although you are more likely to win on some bets (such as "odd", "even", "red", "black"), the payouts for each of these bets is adjusted so that the expected winnings in each case are the same.

It is also possible to make several bets at once. These *combination* bets are a bit more complicated to analyse. For example, a player may play the number 7 straight-up, but also bet *even* numbers at the same time. In order to play both these bets in combination, the player has to make two bets. Let's suppose the player bets $5 on the number 7, and $5 on all even numbers. Note that the numbers "0" and "00" are not considered even numbers. The possible outcomes and their net payouts are:

Outcome	Probability	Net Payout Bet 1	Net Payout Bet 2	Combined Net Payout
(1) The number 7 hits	1/38	175	-5	170
(2) An even number hits	18/38	-5	5	0
(3) Any other number	19/38	-5	-5	-10

There are three possible outcomes (or groups of outcomes) labelled (1), (2), and (3) in the table above. Each outcome has it's own probability ($\mathbf{p_i}$) and a combined net payout ($\mathbf{X_1 - z}$, $\mathbf{X_2 - z}$, **...**) . The expected net value for this particular combination bet is:

$$E(\mathbf{X}_{\text{COMBINATION}} - z) = (\boldsymbol{p_1}\ (\mathbf{X_1 - z})) + (\boldsymbol{p_2}\,(\mathbf{X_2 - z})) + (\boldsymbol{p_3}\ (\mathbf{X_3 - z}))$$
$$\Leftrightarrow \quad E(\mathbf{X}_{\text{COMBINATION}} - z) = [(1/38) \times (170)] + [(18/38) \times (0)] + [(19/38) \times (-10)]$$
$$\Leftrightarrow \quad E(\mathbf{X}_{\text{COMBINATION}} - z) = - \$0.53$$

Again, as with the simple bets, the expected return of this combination bet is - 5.3 % . This means that for every $10 bet, players can expect to lose 53 cents.

<u>Try these</u>:

- Of all the bets for roulette listed above, only one of these (excluding combination bets) does not lead to a - 5.3 % expected return. Can you find it?
  Hint: Try dividing the numbers in the "chances to win" column into 36.
  Which number stands out?

- At the Casino in Gatineau, Quebec, the original roulette tables had only one zero (as is the custom in Europe). However, after many complaints from customers, the casino installed several "American Roulette" tables with both a zero and a double zero. Calculate the expected value of a *straight-up*, *corner* and *odd* bet against a roulette wheel with only one zero. (*Hint: The payout ratios for a European Roulette wheel are the same as in American Roulette, however the probability of "hitting" a particular number on a European wheel is 1 in 37*). Which game provides the higher expected value?

It appears that roulette isn't that complicated after all. Although casinos make several different types of bets available and allow players to make combination bets, these do not affect the expected winnings for participants (except for the "1st Five" bet as pointed out in the practice problem above). While players may believe they have a special system for beating the casino, this is generally an illusion.

By offering complex games, casinos attempt to hide the simple probabilities that are involved. Games such as *black jack* and *let-it-ride poker* are much more complex than roulette. Nevertheless, these games have all been analysed by mathematicians to ensure that in the long run, the owners of the casino will make a profit.

Of course, all of this does not imply that everyone who goes into a casino will come out losing. On any given day, there will be a number of individuals who are lucky and actually make money. However, it is important not to confuse an occasional winning streak with a "system that beats the odds". In the long run, the average casino player will end up losing money. This is why casinos stay in business.

Fair Bets

The basic form of the examples considered so far involves a set of possible outcomes along with their associated probabilities. The examples also involved using the price of the bet (in the case of Roulette) or the ticket price (in the case of a lottery) to calculate the net expected value (that is the expected value less the cost of the bet/ticket).

In all these examples, the net expected value $E(X - z)$ has been negative. However, it is possible for $E(X - z)$ to be positive or exactly equal to zero.

The case in which $E(X - z) = 0$ is of special interest as it is often the break-even point when making decisions. A bet in which $E(X - z) = 0$ is known as a *fair bet*, as the player – on average – neither expects to win nor lose money. Likewise, for a given set of outcomes, the value of z that ensures $E(X - z) = 0$ is known as the *fair bet price*.

We can calculate the fair bet price by setting $E(X - z) = 0$ and solving for z. For instance, returning to the lottery example, the payouts are $X_{WIN} \equiv \$20,000$, $X_{2ND} \equiv \$500$ and $X_{LOSE} \equiv \$0$. The corresponding probabilities are $p_{WIN} = 0.0001$, $p_{2ND} = 0.002$ and $p_{LOSE} = 0.9979$. The net expected value can be calculated using the equation:

$$E(X - z) = p_{WIN} (X_{WIN} - z) + p_{2ND} (X_{2ND} - z) + p_{LOSE} (X_{LOSE} - z)$$

Setting $E(X - z) = 0$ implies that:

$$\begin{aligned}
& 0 = p_{WIN} (X_{WIN} - z) + p_{2ND} (X_{2ND} - z) + p_{LOSE} (X_{LOSE} - z) \\
\Leftrightarrow\quad & 0 = 0.0001 (20000 - z) + 0.002 (500 - z) + 0.9979(0 - z) \\
\Leftrightarrow\quad & 0 = 0.0001 (20000) - 0.0001z + 0.002 (500) - 0.002z + 0.9979(0) - 0.9979z \\
\Leftrightarrow\quad & 0 = 0.0001 (20000) + 0.002 (500) + 0.9979(0) - (0.0001z + 0.002z + 0.9979z) \\
\Leftrightarrow\quad & 0 = 2 + 1 + 0 - (0.0001 + 0.002 + 0.9979)z \\
\Leftrightarrow\quad & 0 = 3 - (1)z \\
\Leftrightarrow\quad & z = 3
\end{aligned}$$

Hence \$3 is the *fair bet price* in the lottery example.

Alternatively, we can use the fact that $E(X - z) = E(X) - z$. Since we previously calculated that $E(X) = 3$, it is straightforward that z must be 3.

Asset Pricing

Another example of where expected value and fair bet calculations are used is in the stock market and other investments. Examples of this type fall under the category of *asset pricing* . The question here is, given an asset that has an uncertain future value – such as shares in a particular company – what price would an individual be willing to pay to purchase these shares?

Asset pricing models can be quite complex. Yet the simple examples provided here illustrate the basic principles. For example, suppose (unrealistically) that it is known that by the start of next week the stocks for ABC company will be worth \$40 per share with 15% probability, \$20 per share with 25% probability, and \$0 per share with 60% probability. Then the outcomes are $X_1 = \$40$, $X_2 = \$20$, and

X_3 = \$0 and the corresponding probabilities are p_1 = 0.15, p_2 = 0.25 and p_3 = 0.60 . (Note: since the timeframe – one week – is extremely short, we are ignoring inflation/present value considerations). Hence, the expected net value for the asset is:

$$E(X - z) = p_1(X_1 - z) + p_2(X_2 - z) + p_3(X_3 - z)$$

where z ≡ the share price in ABC company.

Given this information, we can determine the *fair bet price* of z, by setting E(X - z) = 0 and solving for z. Setting E(X - z) = 0 implies:

$$0 = p_1(X_1 - z) + p_2(X_2 - z) + p_3(X_3 - z)$$
$$\Leftrightarrow \quad 0 = 0.15(40 - z) + 0.25(20 - z) + 0.60(0 - z)$$
$$\Leftrightarrow \quad 0 = 0.15(40) + 0.25(20) + 0.60(0) - (0.15z + 0.25z + 0.6z)$$
$$\Leftrightarrow \quad 0 = 6 + 5 + 0 - (0.15 + 0.25 + 0.6)\, z$$
$$\Leftrightarrow \quad 0 = 11 - (1)\, z$$
$$\Leftrightarrow \quad z = 11$$

Hence z = \$11 is the *fair bet price* for a share in ABC company. Thus, we would not expect an investor to pay more than \$11 per share for ABC company. (Otherwise, the investor would expect to lose money, and the point of investing is to make money.)

We could extend this example as follows. Now suppose that given some new information, the expected prices for the shares change to \$40 per share with 20% probability, \$20 per share with 30% probability and \$0 per share with 50% probability. (That is, X_1 = \$40, X_2 = \$20, X_3 = \$0; p_1 = 0.20, p_2 = 0.30 and p_3 = 0.50). This would alter the expected value for shares of ABC. Using calculations similar to those above, it can be shown that with these new expectations, the fair bet price for ABC will rise to \$14.

It is in this manner that the value of stocks rise and fall. Of course, investors typically do not agree on the possible outcomes or the corresponding probabilities. Investors who value the stock higher than the current price will want to buy shares, and investors who think the stock is overvalued will want to sell shares. It is this constant trading between investors with different expectations that causes stock prices to rise and fall – not only over the long-term – but even from minute-to-minute during the day.

As stated previously, this an over-simplified model of the stock market. Much more sophisticated models are used by financial analysts. But this simple model illustrates a couple of points that are true even within more complex models. First, it shows how expected value calculations play an important role in analysing stock market prices. Moreover, it illustrates how the basic mathematics behind the stock market are similar to the mathematics behind gambling. Paradoxically, gambling is often regarded as morally wrong while investing in the stock market is often seen as financially prudent. Yet as this simple example demonstrates, there are many similarities between the two.